

Scaling Properties of the Lorenz System and Dissipative Nambu Mechanics[†]

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Abstract

In the framework of Nambu Mechanics, we have recently argued that Non-Hamiltonian Chaotic Flows in R^3 , are dissipation induced deformations, of integrable volume preserving flows, specified by pairs of Intersecting Surfaces in R^3 . In the present work we focus our attention to the Lorenz system with a linear dissipative sector in its phase space dynamics. In this case the Intersecting Surfaces are Quadratic. We parametrize its dissipation strength through a continuous control parameter ϵ , acting homogeneously over the whole 3-dim. phase space. In the extended ϵ -Lorenz system we find a scaling relation between the dissipation strength ϵ and Reynolds number parameter r . It results from the scale covariance, we impose on the Lorenz equations under arbitrary rescalings of all its dynamical coordinates. Its integrable limit, ($\epsilon = 0$, fixed r), which is described in terms of intersecting Quadratic Nambu “Hamiltonians” Surfaces, gets mapped on the infinite value limit of the Reynolds number parameter ($r \rightarrow \infty$, $\epsilon = 1$). In effect weak dissipation, through small ϵ values, generates and controls the well explored Route to Chaos in the large r -value regime. The non-dissipative $\epsilon = 0$ integrable limit is therefore the gateway to Chaos for the Lorenz system.

[†]The present work is dedicated to the memory of Prof. J.S.Nicolis

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1 Introduction

In the framework of Nambu mechanics we have recently [1, 2] proposed a geometric approach for the study of dynamical dissipative systems [3]. It is implemented through a special decomposition of the flow of the system in two parts. The non-dissipative and therefore phase-space volume preserving component and the dissipative constituent part which is volume contracting. In the particular case of $D = 3$ phase-space dimensions such a splitting takes the suggestive form

$$\dot{\vec{x}} = \vec{\nabla} H_1 \times \vec{\nabla} H_2 + \vec{\nabla} D \quad (1.1)$$

where H_1, H_2 are the Nambu “Hamiltonians” or equivalently, the Clebsch-Monge potentials [4] where D is the dissipation potential. In particular, for the Lorenz system [5, 6],

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= x(r - z) - y \\ \dot{z} &= xy - bz \end{aligned} \quad (1.2)$$

the functions H_1, H_2, D are:

$$\begin{aligned} H_1 &= \frac{1}{2} (y^2 + (z - r)^2) \\ H_2 &= \sigma z - \frac{x^2}{2} \\ D &= -\frac{1}{2} (\sigma x^2 + y^2 + bz^2) \end{aligned} \quad (1.3)$$

The non-dissipative Lorenz system which was studied carefully in [1] was also investigated some time ago in [7] and in a similar vein further back in time by [8]

$$\dot{\vec{x}} = \vec{\nabla} H_1 \times \vec{\nabla} H_2 \quad (1.4)$$

is an integrable dynamical system, where the trajectories in phase space are given by the intersection of the surfaces defined by the conserved Nambu “Hamiltonians” (H_1, H_2). In this picture the dynamical system in rel(1.1) is defined through its non-dissipative part of eq.(1.4), over which its phase space dissipation $\vec{\nabla} D$ operates. In order to control its dissipation strength and take hold as accurately as possible of its phase space volume contracting effect, we introduce an associated control parameter ϵ as follows:

$$\dot{\vec{x}} = \vec{\nabla} H_1 \times \vec{\nabla} H_2 + \epsilon \vec{\nabla} D \quad (1.5)$$

It is defined in a range of values on the interval $[0, 1]$ with its value $\epsilon = 0$ recovering the Lorenz system nondissipative part, whereas for $\epsilon = 1$ the full system being obtained. Clearly the ϵ -parameter controls the dissipation rate of the phase-space volume of the dynamical system in question. For the Lorenz case, for example,

$$\delta V(t) = e^{-\epsilon(1+\sigma+b)t} \delta V(0) \quad (1.6)$$

We could say that this is nothing else but a rescaling of time. As we shall demonstrate in the particular but very interesting case of the Lorenz system, the rescaling of time is possible only under appropriate simultaneous rescalings of all the dynamical variables x, y, z of the phase space, the time t and more importantly, of the Reynolds number r , while keeping the two others b, σ intact.

At this stage we could observe that the introduction of the control parameter ϵ measures also the strength of the attraction of the Lorenz ellipsoid [5].

Indeed, the rate of change of the Liapunov function

$$V = \frac{1}{2} (rx^2 + \sigma y^2 + \sigma(z - 2r)^2) = \sigma H_1 + r H_2 + \frac{3}{2} r^2 \sigma \quad (1.7)$$

is given by

$$\dot{V} = \dot{\vec{r}} \cdot \nabla V = -\epsilon \sigma [rx^2 + y^2 + b(z - r)^2 - br^2] \quad (1.8)$$

When ϵ is a small positive number, the velocity is entering the Lorenz ellipsoid V , if all of its points are located outside the ellipsoid defined by the vanishing of the expression of the last equation. On the other hand, in the case of the nondissipative system of eq.(1.4), i.e. $\epsilon = 0$ the velocity of the particle is tangent to the ellipsoid $V = V(x_0, y_0, z_0)$ where (x_0, y_0, z_0) are the initial conditions of the motion, because H_1, H_2 are conserved. Thus ϵ controls the overall attractive strength of the attractor.

In our recent work we also introduced a method of Matrix-Heisenberg Quantization for the Lorenz system aiming to examine the compatibility of the classical Lorenz strange attractor

dynamics with the fundamental principles of Quantum Mechanics. We defined rigorously the quantization procedure starting from the non-dissipative system (1.4) where we replaced the phase space coordinates x, y, z with $N \times N$ hermitean matrices $\hat{X}, \hat{Y}, \hat{Z}$ consistently with the appropriate rules of Quantum Correspondence. The quantum dissipation was defined to respect the above quantum correspondence principle. It is intuitively obvious that, if dissipation strength in phase space is big enough, the approach of the system to the classical limit is fast and efficient expressed through the exponentially fast vanishing in time of the commutators of $\hat{X}, \hat{Y}, \hat{Z}$. Thus it is imperative for the study of the coexistence of strange attractors and quantum mechanics long time scales to be maintained in the non-vanishing of quantum commutators. This is feasible through a continuous dissipation strength controlling parameter ϵ .

2 Controlling Dissipation in the Lorenz System

Before we study in detail the scaling properties of the extended Lorenz system (we shall refer to it as the ϵ -Lorenz system)

$$\begin{aligned}\dot{x} &= \sigma(y - \epsilon x) \\ \dot{y} &= x(r - z) - \epsilon y \\ \dot{z} &= xy - \epsilon bz\end{aligned}\tag{2.1}$$

lets analyze the stability properties of its critical points. On this issue we keep in mind that a similar model has been considered in the literature([9, 10, 12]) but with precisely fixed control parameter $\epsilon = \frac{1}{\sqrt{\sigma r}}$, in order that the limit $r \rightarrow \infty$ becomes explicit. For the stability analysis of the critical points we follow [6] and [12].

The critical points of the system of eq. (2.1) are: $P_0 : (x = y = z = 0)$ the origin of the phase space, and $P_{\pm} : (x_{\pm} = \pm\sqrt{b(r - \epsilon^2)}, y_{\pm} = \pm\epsilon\sqrt{b(r - \epsilon^2)}, z_{\pm} = r - \epsilon^2)$. In order that P_{\pm} are real points, we must have $r > \epsilon^2$. Linearization of the system around P_0 gives

$$\frac{d}{dt} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} = \begin{pmatrix} -\epsilon\sigma & \sigma & 0 \\ r & -\epsilon & 0 \\ 0 & 0 & -b\epsilon \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}\tag{2.2}$$

and the eigenvalue problem with

$$\begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} = e^{\lambda t} \begin{pmatrix} \delta x(0) \\ \delta y(0) \\ \delta z(0) \end{pmatrix}\tag{2.3}$$

provides the characteristic Polynomial

$$P_0(\lambda) = -(\lambda + b\epsilon) [(\epsilon\sigma + \lambda)(\epsilon + \lambda) - \sigma r] \quad (2.4)$$

It possesses two negative and one positive eigenvalues. They correspond in turn, to a stable 2-dim. manifold W_0^s with two attracting directions in R^3 as well as to an unstable 1-dim. manifold W_0^u with a single repulsive direction, $\forall \sigma > 0, r > \epsilon^2$.

$$\begin{aligned} \lambda_1 &= -b\epsilon, & \lambda_2 &= -\frac{1}{2} \left[\epsilon(1 + \sigma) + \sqrt{\epsilon^2(1 + \sigma)^2 + 4\sigma(r - \epsilon^2)} \right] \\ \lambda_3 &= \frac{1}{2} \left[\sqrt{\epsilon^2(1 + \sigma)^2 + 4\sigma(r - \epsilon^2)} - \epsilon(1 + \sigma) \right] \end{aligned} \quad (2.5)$$

Linearization, around the P_{\pm} critical points, gives identical characteristic polynomials

$$-P_{\pm}(\lambda) = \lambda^3 + \epsilon(1 + \sigma + b)\lambda^2 + b(r + \sigma\epsilon^2)\lambda + 2\sigma b\epsilon(r - \epsilon^2) \quad (2.6)$$

Since all the coefficients are positive this cubic polynomial has at least one negative eigenvalue(attractive direction in R^3).

The bifurcation from 3-dim stable manifold ($\dim W_{\pm}^s = 3$) to 1-dim stable manifold ($\dim W_{\pm}^s = 1$) occurs, when the real part of the pair of complex conjugate roots $\lambda = \lambda_R \pm i\lambda_I$ crosses the value $\lambda_R = 0$. Simple algebra shows that $\lambda_R = 0$ is equivalent with the statement that the constant term in P_{\pm} in rel.(2.6) is the product of the coefficients of the linear and the quadratic terms. Thus the critical value of r is given by,

$$r_c(\epsilon) = \epsilon^2 \sigma \frac{\sigma + b + 3}{\sigma - b - 1} \quad (2.7)$$

For $r > r_c(\epsilon)$ the subcritical Hopf bifurcation leads to the aperiodic attractive orbits of the famous Lorenz Strange attractor[5]. From ref.(2.7) we get the interesting result that for small ϵ close to the non-dissipative integrable limit of $\epsilon = 0$ the strange attractors appear for small values of $r > r_c(\epsilon)$.

In concluding we present for the critical case of $\lambda_R = 0$, the other roots of the cubic polynomial P_{\pm} . The real root is given by

$$\lambda_0 = -\epsilon(1 + \sigma + b) \quad (2.8)$$

whereas the imaginary component of the complex conjugate one is:

$$\lambda_I = \frac{2\epsilon^2 \sigma(\sigma + 1)}{\sigma - b - 1} \quad (2.9)$$

λ_I represents the angular frequency of the orbit rotation around the critical points P_{\pm} .

In what follows we are going to investigate the scaling properties of the extended ϵ -Lorenz system which will make transparent the ϵ -dependence of $r_c(\epsilon)$ in rel. (2.7).

3 Scaling Relations in the Lorenz Model

Firstly we rescale independently all the variables of the system $\alpha, \beta, \gamma, \lambda \in R$

$$x = \alpha x' , \quad y = \beta y' , \quad z = \gamma z' , \quad t = \lambda t' \quad (3.1)$$

We demand that in the transformed primed system a similar structure of constants appear in the equations of motion, i.e. of the r, ϵ, b, σ type (the dot represents time derivative with respect to t'),

$$\begin{aligned} \dot{x}' &= \sigma'(y' - \epsilon' x') \\ \dot{y}' &= x'(r' - z') - \epsilon' y' \\ \dot{z}' &= x' y' - \epsilon' b' z' \end{aligned} \quad (3.2)$$

we obtain

$$\epsilon' = \lambda \epsilon, \quad b' = b, \quad \sigma' = \sigma, \quad r' = \lambda^2 r, \quad \alpha = \frac{1}{\lambda}, \quad \beta = \gamma = \frac{1}{\lambda^2} \quad (3.3)$$

The dynamical variables x, y, z are related to their primed ones as:

$$x(t) = \frac{1}{\lambda} x'(\lambda t) , \quad y(t) = \frac{1}{\lambda^2} y'(\lambda t) , \quad z(t) = \frac{1}{\lambda^2} z'(\lambda t) \quad (3.4)$$

We will show below that the established covariance property of the ϵ -Lorenz system under the scalings (3.3 , 3.4) implies interesting constraints for the dependence of the phase space coordinates x, y, z on the time, the parameters r, ϵ, σ, b as well as on the initial conditions x_0, y_0, z_0 .

Indeed the time evolution of the phase space coordinates is given by the exponential of the Liouville operator \mathcal{L}_0 , ($x_0 = x(0), y_0 = y(0), z_0 = z(0)$),

$$x(t; r, \epsilon, \sigma, b, x_0, y_0, z_0) = e^{t\mathcal{L}_0} \cdot x_0 \quad (3.5)$$

where

$$\mathcal{L}_0 = \sigma(y_0 - \epsilon x_0) \partial_{x_0} + [x_0(r - z_0) - \epsilon y_0] \partial_{y_0} + (x_0 y_0 - \epsilon b z_0) \partial_{z_0} \quad (3.6)$$

Under the scaling (3.3-3.4) which we rewrite explicitly as

$$t \rightarrow \frac{1}{\lambda} t , \quad \sigma \rightarrow \sigma , \quad b \rightarrow b , \quad r \rightarrow \lambda^2 r , \quad \epsilon \rightarrow \lambda \epsilon \quad (3.7)$$

as well as,

$$x_0 \rightarrow \lambda x_0 , \quad y_0 \rightarrow \lambda^2 y_0 , \quad z_0 \rightarrow \lambda^2 z_0 \quad (3.8)$$

($\forall \lambda \in R$), it is straightforward to check that \mathcal{L}_0 scales as

$$\mathcal{L}_0 \rightarrow \lambda \mathcal{L}_0 \quad (3.9)$$

At last from rel.(3.5) we obtain that $\forall t \in R$,

$$x\left(\frac{1}{\lambda}t; \lambda^2 r, \lambda \epsilon, \sigma, b, \lambda x_o, \lambda^2 y_o, \lambda^2 z_o\right) = \lambda x(t; r, \epsilon, \sigma, b, x_o, y_o, z_o) \quad (3.10)$$

and for the $y(t)$ and $z(t)$ coordinates we obtain respectively

$$y \rightarrow \lambda^2 y, \quad z \rightarrow \lambda^2 z \quad (3.11)$$

We note that the nondissipative Lorenz system ($\epsilon = 0$) satisfies scaling relation (3.10). When ($\epsilon \neq 0$) we impose $\lambda \cdot \epsilon = 1$ on the LHS of eq.(3.10) which implies that the whole $r - \epsilon$ plane is foliated by the continuous set of parabolas

$$\frac{r}{\epsilon^2} = r_{\mathcal{L}} = \text{constant} \quad (3.12)$$

Each parabola corresponds to a different phase of the Lorenz system which is specified by the value of the Reynold's parameter r for which the parabola cuts the $\epsilon = 1$ line . On the other hand if we fix a given parabola all of its points are related through the scaling relations (3.3-3.4). Indeed the condition $\lambda \cdot \epsilon = 1$ implies that:

$$x(t; r, \epsilon, \sigma, b, x_o, y_o, z_o) = \epsilon x\left(\epsilon t; \frac{r}{\epsilon^2}, 1, \sigma, b, \frac{1}{\epsilon}x_o, \frac{1}{\epsilon^2}y_o, \frac{1}{\epsilon^2}z_o\right) \quad (3.13)$$

Similarly for y, z we have

$$y(t; r, \epsilon, \sigma, b, x_o, y_o, z_o) = \epsilon^2 y\left(\epsilon t; \frac{r}{\epsilon^2}, 1, \sigma, b, \frac{1}{\epsilon}x_o, \frac{1}{\epsilon^2}y_o, \frac{1}{\epsilon^2}z_o\right) \quad (3.14)$$

$$z(t; r, \epsilon, \sigma, b, x_o, y_o, z_o) = \epsilon^2 z\left(\epsilon t; \frac{r}{\epsilon^2}, 1, \sigma, b, \frac{1}{\epsilon}x_o, \frac{1}{\epsilon^2}y_o, \frac{1}{\epsilon^2}z_o\right) \quad (3.15)$$

Equations (3.13-3.15) constitute the main result of the present work. It relates the time evolution of the extended ϵ -Lorenz system $\epsilon \neq 1$ with the standard Lorenz one $\epsilon = 1$. Indeed, by denoting with $r_{\mathcal{L}}$ the ratio $r_{\mathcal{L}} = \frac{r}{\epsilon^2}$ we can read off eqs.(3.13-3.15) in two distinct but complementary ways.

Firstly, if we fix $r_{\mathcal{L}}$ we see that the r parameter of the ϵ -Lorenz system (LHS of eqs.(3.13-3.15)) ranges between the values of $0 < r < r_{\mathcal{L}}$ when $0 < \epsilon < 1$. It follows that the physics of all the points on the parabolas $r = r_{\mathcal{L}} \cdot \epsilon^2$ of the ($r - \epsilon$) plane is the same.

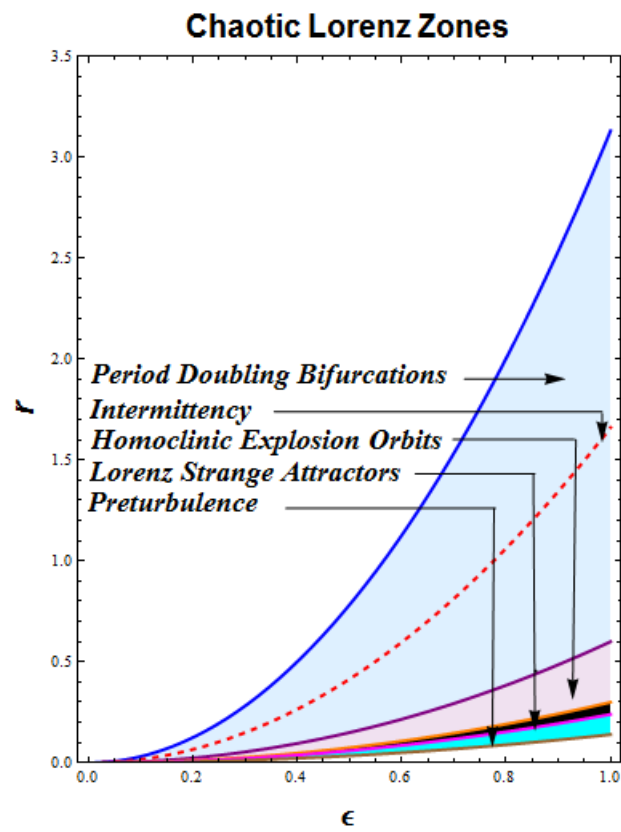


Figure 1: Lorenz System Structure of Dynamics in r - ϵ plane

On the other hand if we fix r for the ϵ -Lorenz system (LHS of (3.13-3.15)), by varying ϵ ($0 < \epsilon < 1$), we are able to scan the region of the standard Lorenz system, for $r_{\mathcal{L}}$, $r < r_{\mathcal{L}} < \infty$ (see fig.1).

Some remarks are in order:

By fixing r in the ϵ -Lorenz system and taking the limit $\epsilon \rightarrow 0$ we recover the non-dissipative Lorenz system of eq.(1.4). It is integrable and describes the motion of a particle in an one dimensional anharmonic potential [1](or the pendulum). We can still rescale this system by choosing $\lambda = \frac{1}{\sqrt{r}}$ and we discover the infinite r limit of the full Lorenz system, which has been studied in detail in ref.[10, 11]. In this limit the Lorenz attractor is degenerate into an 8-figure stable limit cycle (see also in [14]). In the above mentioned works, the $\frac{1}{r}$ correction has been studied and simple bifurcations of the limit cycle have been observed. In [10] an exhaustive search has been made by lowering the values of r . It has been discovered that for relative small values of the 8-figure still survives somewhat deformed nevertheless up to $r = 300$.

In table 1 we summarize the behavior of the Lorenz system for various interval values of the Reynold's number $r_{\mathcal{L}}$ which comprise the famous Feigenbaum Route to Chaos. In the present case of the extended ϵ -Lorenz system with a (r, ϵ) space of parameters they correspond to parabolic zones of equivalent dynamical systems, specified by the scaling relation $r = r_{\mathcal{L}}\epsilon^2$. In table 1 we depict two such characteristic equivalent Lorenz systems for $\epsilon = 1$ and $\epsilon = .01$. The Route to Chaos for each of them is characterised by the following distinct behaviours for different values of r for fixed ϵ :

For values of the $r = r_{\mathcal{L}}$ with $\epsilon = 1$ in the range ($30 < r < 313$) is composed of an infinite cascade of period doubling bifurcations ($2, 2^2, 2^3, \dots, 2^n$) which appear until a critical value of r , $r^c = 30.1$ is reached below which the strange attractors appear for r in the range $24.06 < r < 30.1$. At approximately the value of $r = r_{\mathcal{L}} = 166.07$ we have the Intermittency scenario to Chaos which was first described by Pomeau and Maneville [15]. For $13 < r < 24.06$ an interesting "preturbulence" regime has been observed in the works of J.Yorke et.al.[16]. For a complete and detailed description of the behaviour of the Lorenz system for all values of r one should look at the work of Sparrows who does it exhaustively[6]. All of the above are depicted in figure 1.

Table 1: (r- ϵ) Lorenz Roadmap to Chaos

	$r = r_{\mathcal{L}} , \epsilon = 1$	$r , \epsilon = 0.01$
Period Doubling Bifurcations	$59.5 < r < 313$	$0.00596 < r < 0.0313$
Intermittency	$r = 166.07$	$r = 0.0166$
Orbits from Homoclinic Explosions	$30.1 < r < 59.5$	$0.0030 < r < 0.00596$
Strange Attractors	$24.06 < r < 30.1$	$0.0024 < r < 0.00301$
Preturbulence	$13.9 < r < 24.06$	$0.00139 < r < 0.0024$

4 On the Scale Invariant Lorenz System

In the last part of this work we introduce a form of the ϵ -Lorenz system which is by construction invariant under the scalings in rel.(3.3-3.4). To this end we define new independent and dependent variables($r > \epsilon^2$) :

$$\tau = \sqrt{r - \epsilon^2}t , \quad X = \frac{x}{\sqrt{r - \epsilon^2}} , \quad Y = \frac{y}{r - \epsilon^2} , \quad Z = \frac{z}{r - \epsilon^2} \quad (4.1)$$

which satisfy the system of eqns.

$$\dot{X} = \sigma(Y - \zeta X) , \quad \dot{Y} = X(1 + \zeta^2 - Z) - \zeta Y , \quad \dot{\zeta} = XY - b\zeta Z \quad (4.2)$$

with $\zeta = \frac{\epsilon}{\sqrt{r - \epsilon^2}}$

The derivative “dot” is with respect to the new time τ . Every parabola $\frac{r}{\epsilon^2} = c$ in the $r - \epsilon$ plane is determined by a fixed value of

$$\zeta = \frac{1}{\sqrt{\frac{r}{\epsilon^2} - 1}} \quad (4.3)$$

or

$$r = \left(1 + \frac{1}{\zeta^2}\right) \epsilon^2 \quad (4.4)$$

By construction it is easy to see that the variables τ, X, Y, Z are scale invariant under rel.(3.3-3.4). We may also observe that in the system (4.2) the parameter ζ appears in both places where the Reynolds number as well as the dissipation control parameter ϵ appeared in eq.(2.1) independently.

Repeating the critical point analysis, as before, and their stability we find that there are three critical points. the point $P_o = (0, 0, 0)$ with two attracting and one repelling directions with corresponding eigenvalues:

$$\begin{aligned} \lambda_1 &= -b\zeta , \quad \lambda_2 = -\frac{1}{2}[\zeta(1 + \sigma) + \sqrt{\zeta^2(1 + \sigma^2 + 4\sigma)}] , \\ \lambda_3 &= \frac{1}{2}[\sqrt{\zeta^2(1 + \sigma^2 + 4\sigma)} - \zeta(1 + \sigma)] \end{aligned} \quad (4.5)$$

as well as two symmetric , under the reflection $(x, y, z) \rightarrow (-x, -y, z)$ critical points $P_{\pm} = (\pm\sqrt{b}, \pm\zeta\sqrt{b}, 1)$ with characteristic polynomial

$$P_{\pm}(\lambda) = \lambda^3 + \zeta(1 + b + \sigma)\lambda^2 + b[1 + \zeta^2(1 + \sigma)]\lambda + 2b\zeta\sigma \quad (4.6)$$

Once more the bifurcation condition for the appearance of the strange attractor is $\zeta < \zeta_c$ where

$$\zeta_c^2 = \frac{\sigma - b - 1}{(\sigma + 1)(\sigma + b + 1)} \quad (4.7)$$

For the standard values of $\sigma = 10$ and $b = \frac{8}{3}$ we find

$$\zeta_c^2 = \frac{19}{11 \cdot 41} = \frac{11}{451} \quad (4.8)$$

We note finally that the scale invariant form of the evolution eqs. (4.2) can be cast in a form, which exhibits the role of dissipation on the non-dissipative sector of the system. In figure 2 we plot $r_{\mathcal{L}} = \frac{r}{\epsilon^2}$,i.e. the relation of $r_{\mathcal{L}}$ against ζ . Each value of ζ determines a unique parabola in the r - ϵ parameter plane of the equivalent physics of the Lorenz system.

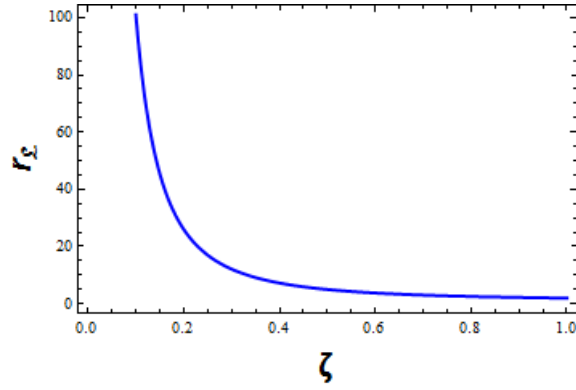


Figure 2: Lorenz ζ Dissipation Parameter

The Non-Dissipative part was defined in rel.(1.4) and corresponds to the $\zeta = 0$ case :

$$\dot{X} = \sigma Y, \quad \dot{Y} = X(1 - Z), \quad \dot{Z} = XY \quad (4.9)$$

We choose to work with the Cylinder intersecting with a Paraboloid as the two Nambu “Hamiltonians” among the whole set of $SL(2, R)$ geometries [1].

$$H_1 = \frac{1}{2} [Y^2 + (1 - Z)^2], \quad H_2 = \sigma Z - \frac{X^2}{2} \quad (4.10)$$

or

$$\dot{\vec{X}} = \nabla H_1 \times \nabla H_2 \quad (4.11)$$

For $\zeta = 0$ the two surfaces H_1, H_2 are conserved

$$H_1 = H_1(t = 0) \quad , \quad H_2 = H_2(t = 0) \quad (4.12)$$

and they define by their intersection the trajectory of the system (4.9). When $\zeta \neq 0$, H_1, H_2 are not conserved. Still one can use in the place of X, Y, Z the variables X and H_2 in order to elucidate what happens. For $\zeta \neq 0$ we obtain [11]

$$\ddot{X} + (1 + \sigma)\zeta\dot{X} + X \left[\frac{X^2}{2} + H_2 - \sigma \right] = 0 \quad (4.13)$$

and

$$\dot{H}_2 + \zeta \left[bH_2 - \sigma(1 - \frac{b}{2\sigma})X^2 \right] = 0 \quad (4.14)$$

We see now the crucial role of ζ in the introduction of two terms: Firstly a friction term proportional to \dot{X} and secondly, a memory term in the anharmonic potential minimum. By eliminating H_2 and using rel.(4.14) we obtain the Takeyama evolution memory term, which changes in a non-Markovian random manner the symmetry of the single well potential into the double well one [17].

5 Conclusions-Open Problems

The introduction of the ϵ -Lorenz system, which controls the strength of the dissipative sector implies the existence of a weak (under-dissipated) and strong(over-dissipated) phases for the Lorenz system. They are separated in the $r - \epsilon$ parameter space phase diagram by the critical line $\epsilon = 1$ which reproduces for different values of r the route to chaos of the Lorenz system, which is presented so lucidly in the work of Sparrow[6]. We have demonstrated therein that a Lorenz system with weak dissipation (small ζ values) is equivalent to the one with large values of the Reynolds number r for the standard Lorenz system with $\epsilon = 1$.

The exploration of the scaling properties though, brought unexpected new information about the ϵ Lorenz which smoothly joins to the $\epsilon = 0$ integrable case. The scaling eqs.(3.3-3.4) show that we have very specific combinations of the time, the initial conditions and the parameters r and ϵ in each order in the Taylor expansion of the solution with respect to time, such that the scaling relation holds true. Also the bifurcation behaviour of the solution can be controlled by the one parameter $\zeta = \frac{\epsilon}{\sqrt{r-\epsilon^2}}$ in rel.(4.3) which parametrizes the foliation of the r - ϵ plane by one parameter family of parabolas. All the points of each parabola are physically equivalent. Last but not least, the Nambu surfaces being the appropriate geometrical tool for the $\zeta = 0$ integrable case, are useful also for the small ζ range which corresponds to the large r model ($\epsilon = 1$) where we know that successive bifurcations of the figure 8 periodic limit cycle [10, 11] lead to the strange attractor configurations. The Non-dissipative Limit is thus the gateway to Chaotic and turbulent Flows for the Lorenz system. Interestingly, it may be also the gateway to Quantum Chaos in a matrix formulation of the ϵ -Lorenz system, where the expected presence of decoherence can become suppressed in an controllable way. This interesting possibility will be investigated in a separate work in the near future.

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